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Constant boundary-value problems for p-harmonic maps with potential

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Abstract

In this paper, after introducing a large class of manifolds which includes the manifolds with strictly negative curvature bounded between two negative constants as special cases, we study the constant boundary-value problems of p-harmonic maps with potential defined on such a class of manifolds, and obtain a Liouville-type theorem. The main theorem generalizes that of Karcher and Wood [H. Karcher, J.C. Wood, Non-existence results and growth properties for harmonic maps and forms, J. Reine. Angew. Math. 353 (1984) 165–180] and Chen [Q. Chen, Stability and constant boundary-value problems of harmonic maps with potential, J. Aust. Math. Soc. (Series A) 68 (2000) 145–154] even for the case of the usual harmonic maps or harmonic maps with potential. It can also be applied to the static Landau–Lifshitz equations. Then, using the technique developed there, we prove a Liouville theorem for p-harmonic maps with finite p-energy or slowly divergent p-energy, which answers partially Sampson's conjecture in a more general case.
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1. Introduction and main results

By B^m we denote the unit ball in $\mathbb{R}^m$ with boundary ∂B^m, and by N any Riemannian manifold. Karcher and Wood [7] proved that any harmonic map $u : B^m \to N (m \geq 3)$ which is constant on ∂B^m must be a constant in B^m. Chen [1] obtain the same conclusion in the following more general case.

Let M be an m-dimensional Riemannian manifold with sectional curvature K_M, $-a^2 \leq K_M \leq -b^2$, where a, b are positive constants and $(m-1)b/2 \geq a$. $u : M \to N$ is a harmonic map with potential H, i.e. the critical point of the energy integral

$$E_H(u) = \int_M [e(u) - H(u)],$$

where H is a smooth function on N and $e(u) = \frac{1}{2}|\mathrm{d}u|^2$ is the energy density of u.

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Harmonic maps with potential introduced by Fardoun and Ratto [3] are a new kind of generalized harmonic map and also include the Landau–Lifshitz equations as a special case. In this paper, we will study the constant boundary-value problems for p-harmonic maps with potential from a large class of manifolds which includes the manifolds with strictly negative curvature bounded between two negative constants as special cases.

In order to state our main results, we introduce the following notation for the manifolds satisfying the condition (C) and p-harmonic maps with potential.

Let M^m be complete, simply connected Riemannian manifold with non-positive sectional curvature. For any fixed $x_0 \in M$, $r(x)$ denotes the distance function from x_0 to x, $B_r(x_0)$ stands for the geodesic ball with radius r and center at x_0. Taking an orthonormal frame field $\left\{e_i, \frac{\partial}{\partial r}\right\}$, $i = 1, \ldots, m-1$, around any $x \in \partial B_r(x_0)$, then $e_i \in T_x(\partial B_r(x_0))$. Denote by $\lambda_1, \ldots, \lambda_{m-1}$ the eigenvalues of the radial Ricci operator

$$R\left(\frac{\partial}{\partial r}, \cdot\right)\frac{\partial}{\partial r} : X \in T_x(\partial B_r(x_0)) \mapsto R\left(\frac{\partial}{\partial r}, X\right)\frac{\partial}{\partial r} \in T_x(\partial B_r(x_0)),$$

where R is the curvature operator on M. With this notation, we introduce the following pointwise condition:

$$\underline{\textbf{Condition(C)}} \quad \lambda_i \geq \sum_{j \neq i} \lambda_j, \quad \text{for all indices } i,\ 1 \leq i \leq m-1,$$

or equivalently, for an orthonormal frame $\left\{e_i, \frac{\partial}{\partial r}\right\}$,

$$\left\langle R\left(\frac{\partial}{\partial r}, e_i\right)\frac{\partial}{\partial r}, e_i\right\rangle - \sum_{j \neq i}\left\langle R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r}, e_j\right\rangle \geq 0, \tag{1.1}$$

$\forall\, i = 1, \ldots, m-1$, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product on M.

If the Condition (C) is satisfied at every point of M, we call M a *Riemannian manifold satisfying the condition (C)*. There exists a large class of such manifolds, for example:

(1) Euclidean space $\mathbb{R}^m$ and hyperbolic space $\mathbb{H}^m$.

(2) The Riemannian manifold M^m with sectional curvature K_M bounded as $-a^2 \leq K_M \leq -b^2$, where a, b are positive constants and $\sqrt{m-2}\,b - a \geq 0$.

(3) The Riemannian manifold M^m with sectional curvature K_M satisfying $-a^2 \leq K_M \leq 0$ and the Ricci curvature $\mathrm{Ric}^M \leq -b^2$, where a, b are positive constants and $b > \sqrt{2}\,a$.

Let $u : M \to N$ be a smooth map between Riemannian manifolds M and N, H be a smooth function on N. For $p \geq 2$, we call u a *p-harmonic map with potential H* or for short a *p-H-harmonic map* if it is a critical point of the p-H-energy:

$$E_{p,H}(u) = \frac{1}{p}\int_M |\mathrm{d}u|^p - \int_M H \circ u. \tag{1.2}$$

That is, u is a p-H-harmonic map if and only if

$$\left.\frac{\mathrm{d}E_{p,H}(u_t)}{\mathrm{d}t}\right|_{t=0} = 0,$$

for any one-parameter family of maps $u_t : M \to N$ with $u_0 = u$. Note that, if H is constant, a p-H-harmonic map is called a p-harmonic map. In particular, 2-harmonic maps are just the usual harmonic ones.

We can derive the first variation formula of p-H-harmonic maps in a similar way to those for harmonic maps as follows:

$$\left.\frac{\mathrm{d}E_{p,H}(u_t)}{\mathrm{d}t}\right|_{t=0} = -\int_M \langle \tau_{p,H}(u), V\rangle, \tag{1.3}$$

where $\tau_{p,H}(u) = \tau_p(u) + \operatorname{grad} H \circ u$, $\tau_p(u) = -\mathrm{d}^*(|\mathrm{d}u|^{p-2}\mathrm{d}u)$ and $V = (\mathrm{d}u_t/\mathrm{d}t)|_{t=0}$ is a given vector field along u. Therefore, the Euler–Lagrange equation of $E_{p,H}$ is

$$\tau_{p,H}(u) = 0. \tag{1.4}$$

We can now state our main results in this paper.

Theorem 1. *Let (M^m, g) be a complete, simply connected Riemannian manifold of dimension m $(> p)$ with non-positive sectional curvature K_M and satisfying the condition* (C) *globally. Assume that $u : M \to N$ is a $p-H$-harmonic map such that $u|_{\partial B_r(x_0)} \equiv Q \in N$. If $H(Q) = \max_{y\in N} H(y)$, then u must be constant in $B_r(x_0)$.*

Remark 1. On one hand, let $M = \mathbb{R}^m$ $(m > 2)$, $N = S^2$, $H(y) = H_0 \cdot y$, $y \in S^2$ (where $H_0 \neq 0$ is a constant vector in $\mathbb{R}^3$ and "$\cdot$" denotes the inner product in $\mathbb{R}^3$). Then Theorem 1, for $p = 2$, leads to a conclusion for the static Landau–Lifshitz equations. In particular, if $m = 3$, it is just the result of Hong [6] which asserted that the static Landau–Lifshitz equation with constant boundary value $H_0/|H_0|$ has only a constant solution.

On the other hand, since the manifolds discussed in Theorem 1 include ones with strictly negative curvature bounded between two negative constants, so Theorem 1 recovers that of [1,6] as special cases. Our presentation gives a unified treatment of both the harmonic maps with potential and the static Landau–Lifshitz equations under a more general frame.

With the same class of manifolds as in Theorem 1, and the technique developed in the proof of Theorem 1, we also obtain the following Liouville-type theorem for p-harmonic maps.

Theorem 2. *Let M^m be as in Theorem 1. Assume that $u : M \to N$ is a p-harmonic map with finite p-energy or slowly divergent p-energy. Then u must be constant.*

We say that the p-energy $E_p(u)$ of u, defined by $E_p(u) = \int_M e_p(u)$ and $e_p(u) = \frac{1}{p}|\mathrm{d}u|^p$ called the p-energy density, is slowly divergent if there exists a positive function $\psi(t)$ with $\int_{R_0}^{\infty} \frac{\mathrm{d}t}{t\psi(t)} = \infty$ $(R_0 > 0)$ such that

$$\lim_{R\to\infty} \int_{B_R(x_0)} \frac{e_p(u)}{\psi(r(x))} < \infty. \tag{1.5}$$

Remark 2. When $p = 2$, Theorem 2 answers partially Sampson's conjecture: any harmonic map $u : M \to N$ with finite energy, from a complete simply connected Riemannian manifold of dimension m (≥ 3) with non-positive sectional curvature, must be constant. Concerning this conjecture, Theorem 2 may be viewed as a generalization of results due to Sealey [9], Zhou [12] and Xin [10,11].

2. Some lemmas

Let ∇ be the Levi-Civita connection on M. For an orthonormal frame field $\{e_i, \frac{\partial}{\partial r}\}$, put

$$h_{ij} := \mathrm{Hess}(r)(e_i, e_j) = \left\langle \nabla_{e_i} \frac{\partial}{\partial r}, e_j \right\rangle, \tag{2.1}$$

where $\mathrm{Hess}(r)$ is the Hessian of the distance function r, and $-h_{ij}$ are the coefficients of the second fundamental form of $\partial B_r(x_0)$. The following is the key lemma in the proof of Theorems 1 and 2.

Lemma 3. (1) *Let $(M^m, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold with non-positive sectional curvature K_M and $m \geq 3$; then there exists $\epsilon > 0$ such that when $r \leq \epsilon$, for any $x \in \partial B_r(x_0)$, the following hold for all $i = 1, \ldots, m-1$:*

$$\sum_{j\neq i} h_{jj} - h_{ii} \geq 0. \tag{2.2}$$

(2) *Furthermore, if M is a Riemannian manifold satisfying the condition* (C) *defined before, then* (2.2) *is always true on $\partial B_r(x_0)$ for any $r > 0$.*

Proof. (1) Since $K_M \leq 0$, the Hessian comparison theorem [4] says that, for any $x \in \partial B_r(x_0)$ and $i = 1, \ldots, m-1$,

$$h_{ii} \geq \frac{1}{r}. \tag{2.3}$$

Now, for any $R_0 > 0$, since $B_{R_0}(x_0)$ is compact, we can assume that, on $B_{R_0}(x_0)$, the sectional curvature of M satisfies $-a^2 \le K_M \le -b^2$, where $a > 0$ and $b \ge 0$ are constants. It follows again from comparison theorem [5] that, for all $l = 1, \ldots, m-1$,

$$rh_{ll} \le (ar)\coth(ar). \tag{2.4}$$

Then, from (2.3) and (2.4), we have

$$r\left(\sum_{j\neq i} h_{jj} - h_{ii}\right) \ge (m-2) - (ar)\coth(ar). \tag{2.5}$$

Since $\lim_{r\to 0}(ar)\coth(ar) = 1 \le m-2$, by the continuity, there exists $\epsilon > 0$ such that when $r \le \epsilon$, for any $x \in \partial B_r(x_0)$, we have

$$r\left(\sum_{j\neq i} h_{jj} - h_{ii}\right) \ge 0,$$

which implies (2.2).

(2) Taking an orthonormal frame $\left\{e_i, \frac{\partial}{\partial r}\right\}$ at $x \in \partial B_r(x_0)$ and parallel translation in the radial direction, we obtain an orthonormal frame field so that $\nabla_{\frac{\partial}{\partial r}} e_i = 0,\ \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0$. Then for all indices $l = 1, \ldots, m-1$, a direct computation yields

$$\frac{\mathrm{d}h_{ll}}{\mathrm{d}r} = -\sum_{j=1}^{m-1} h_{lj}h_{jl} - \left\langle R\left(\frac{\partial}{\partial r}, e_l\right)\frac{\partial}{\partial r}, e_l\right\rangle. \tag{2.6}$$

It follows from (2.6) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\sum_{j\neq i} h_{jj} - h_{ii}\right) &= \sum_{j=1}^{m-1} h_{ij}h_{ji} - \sum_{k\neq i}\sum_{l=1}^{m-1} h_{kl}h_{lk} + \left\langle R\left(\frac{\partial}{\partial r}, e_i\right)\frac{\partial}{\partial r}, e_i\right\rangle - \sum_{j\neq i}\left\langle R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r}, e_j\right\rangle \\ &= h_{ii}^2 - \sum_{k,l\neq i} h_{kl}^2 + \left\langle R\left(\frac{\partial}{\partial r}, e_i\right)\frac{\partial}{\partial r}, e_i\right\rangle - \sum_{j\neq i}\left\langle R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r}, e_j\right\rangle. \end{aligned} \tag{2.7}$$

Now take a new basis of the orthogonal complement $\{e_j, j \neq i\}$ of e_i such that the matrix $(h_{kl}),\ k, l \neq i$, can be diagonalizable. Notice that $\sum_{j\neq i} h_{jj}$ is invariant under the change of the basis and for all l, $1 \le l \le m-1$, $h_{ll} > 0$. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\sum_{j\neq i} h_{jj} - h_{ii}\right) &= h_{ii}^2 - \sum_{j\neq i} h_{jj}^2 + \left\langle R\left(\frac{\partial}{\partial r}, e_i\right)\frac{\partial}{\partial r}, e_i\right\rangle - \sum_{j\neq i}\left\langle R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r}, e_j\right\rangle \\ &\ge h_{ii}^2 - \left(\sum_{j\neq i} h_{jj}\right)^2 + \left\langle R\left(\frac{\partial}{\partial r}, e_i\right)\frac{\partial}{\partial r}, e_i\right\rangle - \sum_{j\neq i}\left\langle R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r}, e_j\right\rangle. \end{aligned} \tag{2.8}$$

Setting $g_i(r) = \sum_{j\neq i} h_{jj} - h_{ii}$ and $A(r) = \sum_{l=1}^{m-1} h_{ll}$, with the same ϵ (>0) as obtained in Lemma 3(1), from (2.8) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(g_i e^{\int_\epsilon^r A\mathrm{d}r}\right) \ge e^{\int_\epsilon^r A\mathrm{d}r} \cdot \left(\left\langle R\left(\frac{\partial}{\partial r}, e_i\right)\frac{\partial}{\partial r}, e_i\right\rangle - \sum_{j\neq i}\left\langle R\left(\frac{\partial}{\partial r}, e_j\right)\frac{\partial}{\partial r}, e_j\right\rangle\right). \tag{2.9}$$

Since M satisfies the condition (C) globally, i.e. (1.1) holds at all points of M, so we obtain the following monotonicity inequality:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(g_i e^{\int_\epsilon^r A\mathrm{d}r}\right) \ge 0,$$

which implies that, for any $R > \epsilon$,

$$g_i(R)e^{\int_\epsilon^R A\mathrm{d}r} \geq g_i(\epsilon) \geq 0.$$

In particular, $g_i(R) \geq 0$ for any $R > \epsilon$. Combining this with part (1) above, we complete the proof of the lemma. □

The next lemma is very useful in the proof of Theorem 2. Let $u : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds M and N with Riemannian metrics g, h respectively. Define the stress p-energy tensor by

$$S_p(u) = e_p(u)g - |\mathrm{d}u|^{p-2}u^*h,$$

then, for any smooth tangent vector field X on M, a straightforward calculation gives

$$(\mathrm{div}\, S_p(u))(X) = -\langle \tau_p(u), \mathrm{d}u(X)\rangle, \tag{2.10}$$

which implies that if u is a p-harmonic map, then $\mathrm{div}\, S_p(u) = 0$.

Lemma 4. *Let $D \subset M$ be a compact domain such that ∂D is a smooth hypersurface in M, and* $\mathbf{n}$ *the outer unit normal vector of ∂D; then for any smooth tangent vector field X with compact support,*

$$\int_{\partial D} e_p(u)\langle X, \mathbf{n}\rangle = \int_{\partial D} |\mathrm{d}u|^{p-2}\langle \mathrm{d}u(X), \mathrm{d}u(\mathbf{n})\rangle + \int_D (\mathrm{div}\, S_p(u))(X) + \int_D \langle S_p(u), \nabla X\rangle, \tag{2.11}$$

where $\nabla X(V, W) = \langle \nabla_X V, W\rangle$.

Proof. By a standard computation, we obtain the following equalities:

$$\mathrm{div}(e_p(u)X) = \nabla_X e_p(u) + e_p(u)\langle \nabla_{e_i} X, e_i\rangle,$$
$$\nabla_X e_p(u) = \mathrm{div}(|\mathrm{d}u|^{p-2}\langle \mathrm{d}u(X), \mathrm{d}u(e_i)\rangle e_i) - \langle \mathrm{d}u(X), \tau_p(u)\rangle - |\mathrm{d}u|^{p-2}\langle \nabla X, u^*h\rangle.$$

Hence we have

$$\mathrm{div}(e_p(u)X) = \mathrm{div}(|\mathrm{d}u|^{p-2}\langle \mathrm{d}u(X), \mathrm{d}u(e_i)\rangle e_i) - \langle \mathrm{d}u(X), \tau_p(u)\rangle + \langle S_p(u), \nabla X\rangle.$$

Since Supp$X$ is compact, by applying Green's formula to the previous formula and using (2.10), we have the desired formula (2.11). □

In order to simplify the proof of Theorems 1 and 2, we give the following lemma.

Lemma 5. *With the same assumption on M as in Theorem 1, let $u : M \to N$ be a smooth map and $X = r\frac{\partial}{\partial r}$. Then there exists some constant $\delta > 0$ such that*

$$\langle S_p(u), \nabla X\rangle \geq \delta e_p(u). \tag{2.12}$$

Proof. By the definition of $S_p(u)$, a directly computation yields

$$\begin{aligned}\langle S_p(u), \nabla X\rangle &= e_p(u)[1 + rh_{jj}] - |\mathrm{d}u|^{p-2}\left\{\left|\mathrm{d}u\left(\frac{\partial}{\partial r}\right)\right|^2 + rh_{jk}\langle \mathrm{d}u(e_j), \mathrm{d}u(e_k)\rangle\right\}\\ &= \frac{1}{p}|\mathrm{d}u|^{p-2}\left|\mathrm{d}u\left(\frac{\partial}{\partial r}\right)\right|^2\left(1 + r\sum_{j=1}^{m-1} h_{jj} - p\right)\\ &\quad + \frac{1}{p}|\mathrm{d}u|^{p-2}\sum_{i=1}^{m}|\mathrm{d}u(e_i)|^2\left(1 + r\sum_{j\neq i} h_{jj} - (p-1)rh_{ii}\right)\\ &= \text{(I)} + \text{(II)},\end{aligned} \tag{2.13}$$

where we set

$$\text{(I)} = \frac{1}{p}|\mathrm{d}u|^{p-2}\left|\mathrm{d}u\left(\frac{\partial}{\partial r}\right)\right|^2\left(1 + r\sum_{j=1}^{m-1} h_{jj} - p\right),$$

$$(\mathrm{II}) = \frac{1}{p}|\mathrm{d}u|^{p-2}\sum_{i=1}^{m}|\mathrm{d}u(e_i)|^2\left(1 + r\sum_{j\neq i}h_{jj} - (p-1)rh_{ii}\right).$$

In the following, we will estimate two parts (I) and (II) separately. Firstly, for $m > p$ and $p \in [2, \infty)$, it follows easily from (2.3) that

$$(\mathrm{I}) \geq \frac{m-p}{p}|\mathrm{d}u|^{p-2}\left|\mathrm{d}u\left(\frac{\partial}{\partial r}\right)\right|^2. \tag{2.14}$$

In order to estimate part (II), since we assumed that M satisfies the condition (C), that is to say,

$$\lambda_i \geq \sum_{j\neq i}\lambda_j \geq \sum_{j\neq i}\lambda_j/(p-1),$$

then we know from Lemma 3(2), for $i = 1, \ldots, m-1$, the following inequality holds on the whole of M:

$$\sum_{j\neq i}h_{jj} - h_{ii} \geq 0.$$

Hence,

$$(\mathrm{II}) \geq \frac{1}{p}|\mathrm{d}u|^{p-2}\sum_{i=1}^{m-1}|\mathrm{d}u(e_i)|^2. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13), we complete the proof of Lemma 5. □

3. Proof of the main theorems

Proof of Theorem 1. Since u is a p-H-harmonic map, or equivalently, $\tau_{p,H}(u) = \tau_p(u) + \operatorname{grad} H \circ u = 0$, it follows from (2.10) that

$$\operatorname{div} S_p(u) = \langle \operatorname{grad} H \circ u, \mathrm{d}u\rangle.$$

Then by setting $D = B_R(x_0)$, $X = R\frac{\partial}{\partial r}$ and $\mathbf{n} = \frac{\partial}{\partial r}$, and substituting these into (2.11), we get

$$R\int_{\partial B_R(x_0)} e_p(u) = R\int_{\partial B_R(x_0)}\left|\mathrm{d}u\left(\frac{\partial}{\partial r}\right)\right|^2 + \int_{B_R(x_0)} r\frac{\partial(H\circ u)}{\partial r} + \int_{B_R(x_0)}\langle S_p(u), \nabla X\rangle. \tag{3.1}$$

Since u is constant at $\partial B_R(x_0)$, using (2.12) and (3.1), we have

$$\int_{B_R(x_0)} r\frac{\partial(H\circ u)}{\partial r} + \delta\int_{B_R(x_0)} e_p(u) \leq 0. \tag{3.2}$$

Since $K_M \leq 0$, using Ding's Laplacian comparison theorem [2], we have

$$\triangle r \geq \frac{1}{r}.$$

On the other hand, due to Li [8], we know that

$$\triangle r = \frac{\partial J(\theta, r)}{\partial r}\frac{1}{J(\theta, r)},$$

where $J(\theta, r)\mathrm{d}\theta\mathrm{d}r$ is the volume element of $B_R(x_0)$ in polar coordinates (θ, r) around x_0. Those two facts lead to

$$\frac{\partial}{\partial r}(rJ(\theta, r)) \geq 2J(\theta, r) > 0,$$

from which we have the following integral inequality:

$$\int_0^R r\frac{\partial(H\circ u)}{\partial r}J(\theta, r)\mathrm{d}r = RJ(\theta, R)H(Q) - \int_0^R H\circ u(\theta, r)\frac{\partial}{\partial r}(rJ(\theta, r))\mathrm{d}r$$

$$
\begin{aligned}
&\geq RJ(\theta, R)H(Q) - H(Q)\int_0^R \frac{\partial}{\partial r}(rJ(\theta, r))\mathrm{d}r \\
&= 0.
\end{aligned}
$$

Hence,

$$\int_{B_R(x_0)} r\frac{\partial(H\circ u)}{\partial r} = \int_{\partial B_R(x_0)}\left(\int_0^R r\frac{\partial(H\circ u)}{\partial r}J(\theta, r)\mathrm{d}r\right)\mathrm{d}\theta \geq 0. \tag{3.3}$$

From (3.2) and (3.3), we immediately conclude that $e_p(u) \equiv 0$ in $B_R(x_0)$, namely, u is constant in $B_R(x_0)$, which completes the proof of Theorem 1. □

Proof of Theorem 2. With the assumption on M as in Theorem 2, when u is a p-harmonic map, we have from (2.11) and (2.12) that

$$R\int_{\partial B_R(x_0)} e_p(u) \geq \delta\int_{B_R(x_0)} e_p(u). \tag{3.4}$$

Now suppose that u is a nonconstant map, that is to say the p-energy density $e_p(u)$ does not vanish everywhere, so there exists $R_0 > 0$ such that for $R > R_0$,

$$\int_{B_R(x_0)} e_p(u) \geq C_0, \tag{3.5}$$

where C_0 is a positive constant. Thus (3.4) and (3.5) imply that

$$\int_{\partial B_R(x_0)} e_p(u) \geq \frac{\delta C_0}{R}, \quad \text{for } R > R_0, \tag{3.6}$$

which leads to

$$E_p(u) > \int_{B_R(x_0)} e_p(u) = \int_0^R\left(\int_{\partial B_R(x_0)} e_p(u)\right)\mathrm{d}r \geq \int_{R_0}^R\left(\int_{\partial B_R(x_0)} e_p(u)\right)\mathrm{d}r \geq \int_{R_0}^R \frac{\delta C_0}{r}\mathrm{d}r = \delta C_0 \ln\frac{R}{R_0}.$$

Letting $R \to \infty$, this contradicts the assumption of the finite p-energy, therefore u is constant.

In the case of the slowly divergent p-energy, (3.6) also leads to

$$\lim_{R\to\infty}\int_{B_R(x_0)} \frac{e_p(u)}{\psi(r(x))} = \int_0^\infty \frac{\mathrm{d}r}{\psi(r)}\int_{\partial B_R(x_0)} e_p(u) \geq \delta C_0\int_0^\infty \frac{\mathrm{d}r}{r\psi(r)} \geq \delta C_0\int_{R_0}^\infty \frac{\mathrm{d}r}{r\psi(r)} = \infty,$$

which contradicts (1.5), therefore u is constant. □

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